Int. J. Heat Mass Transfer. Vol. 17, pp. 37-49. Pergamon Press 1974. Printed in Great Britain

THERMAL ANALYSIS OF CYCLIC CRYOGENIC REGENERATORS

M. F. MODEST and C. L. TIEN
Department of Mechanical Engineering, University of California, Berkeley, California, U.S.A.

(Received 6 March 1973 and in revised form 18 June 1973)

Abstract—The system of partial differential equations, together with their boundary conditions, has been established to describe the performance of regenerators subjected to cycling flows, including commonly neglected effects such as internal energy changes of the fluid due to pressure cycling and longitudinal matrix conduction. Exact solutions are obtained for the case of an infinitely large matrix heat capacity. For the case of finite matrix heat capacity the method of perturbations is employed, and the solutions can be considered exact throughout the regenerator except for the regions near the boundaries. In general, the results are contained in a set of three coupled ordinary differential equations, which must be solved numerically. For the important case of negligible matrix heat conduction, however, a closed-form solution is presented here.

NOMENCLATURE

A,	regenerator cross-section;
c_m,	specific heat of matrix material;
C,	constant defined in equation (58);
C_1,	parameter defined in equation (10);
C_2,	integration constant;
h,	gas enthalpy;
h_t,	heat transfer coefficient;
H,	non-dimensional heat transfer coefficient, equation (10);
k,	thermal conductivity;
K,	heat transfer coefficient at cold end;
K_m,	parameter defined by equation (10);
l,	regenerator length;
m,	non-dimensional mass flux;
$\dot{m}$,	mass flux;
M_v,	incomplete gamma function, equation (67);
n, n_1, n_2,	exponents for heat transfer coefficient;
p,	gas pressure;
P,	non-dimensional gas pressure;
Pr,	Prandtl number;
q,	integrated mass flux at cold end;
Q_m,	parameter defined by equation (10);
R,	gas constant;
Re,	Reynolds number;
St,	Stanton number;
t,	time;
t, t_1, t_2,	first order temperature perturbation;
T,	temperature;
u,	internal energy;
V_r,	regenerator void volume, ϵAl;
V_{CE},	expansion volume outside cold end;
w,	average gas velocity;
x,	axial distance;
z,	non-dimensional distance;
α,	specific surface;
β,	constant defined in equation (66);
γ,	ratio of specific heats;
δ,	non-dimensional length of compression period;
ϵ,	regenerator porosity;
η,	regenerator effectiveness;
θ,	non-dimensional time;
ϑ,	second order temperature perturbation;
ν,	constant defined in equation (66);
π,	time span of regenerator cycle;
ρ,	density;
τ,	non-dimensional temperature.

Subscripts

m,	matrix;
$0, l$,	values at cold end ($x = 0$), hot end ($x = l$);
H, L,	values at end of compression (high pressure), and of expansion (low pressure).

Superscript

$^-$,	(bar) integrated value over cycle.

1. INTRODUCTION

SINCE the first thermal regenerator was applied to the production of low temperatures at the end of last century, their use has become more popular. Due to their superior effectiveness and compactness compared to the conventional heat exchangers, they constitute the crucial elements in modern refrigeration devices used for the cooling of superconductors, infrared detectors, etc. While the concept of thermal regenerators is simple, the theory pertaining to the thermal performance is extraordinarily involved. Since the efficiency of a well designed regenerator is high compared to the efficiency of the total refrigeration cycle, little effort had been made until recently to predict its performance accurately. At very low low temperatures, however, the refrigeration cycles become so ineffective that losses due to the regenerator must be minimized. With the increasing application of refrigeration at very low temperatures, more effort has been devoted to the study of sophisticated theories by various investigators, for example [1–4]. All these investigations, however, are limited to the examination of one particular effect or another, neglecting other important factors.

A notable advance towards a rigorous theory has been made by Rea and Smith [5], improving earlier analyses by taking proper account of variable density, mass flux and heat transfer coefficient. Recently this theory has been extended by Modest and Tien [6] to include real-gas and matrix-conduction effects, which become particularly important at very low temperatures. The major shortcomings of these analyses are their approximate nature and the fact that they deal only with time-averaged properties. It is the purpose of this work to improve these analyses by presenting some exact analytical solutions to the governing equations. These solutions not only show in what situations the approximate analyses are applicable, but also provide some physical insight into the variation of temperatures and mass flux with time.

2. GOVERNING EQUATIONS

For the development of the governing equations the following basic assumptions are made: (1) The gas behaves like an ideal gas, (2) Pressure drop along the regenerator is negligibly small, (3) Thermal conduction through the gas in axial direction is negligible, and (4) Wall effects and matrix inhomogenuity are insignificant (i.e. the problem is one-dimensional). Under these conditions the governing equations are [5, 6]:

Gas energy:

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho w h) + \frac{\alpha}{\epsilon} h_t (T - T_m) = 0 \tag{1}$$

Matrix energy:

$$\rho_m \frac{\partial u_m}{\partial t} - \frac{\partial}{\partial x}\left(k_m \frac{\partial T_m}{\partial x}\right) - \frac{\alpha}{1-\epsilon} h_t (T - T_m) = 0 \tag{2}$$

Gas continuity:

$$\frac{\partial}{\partial x}(\rho w) + \frac{\partial \rho}{\partial t} = 0. \tag{3}$$

The first terms in equations (1) and (2) describe the change of internal energy of gas and matrix. The second term in equation (1) stands for the change of enthalpy of the gas flowing through the control volume, while the second term in equation (2) denotes the conduction along the matrix. The last terms in both equations give the convective energy exchange between gas and matrix inside the control volume.

In general, the mass flux can be calculated at the cold end ($x = 0$) and the temperatures of the gas entering the matrix are known, so that the boundary conditions are:

cold end $x = 0$:

$$\dot m(0, t) = \epsilon A \rho w(0, t) = \dot m_0(t)$$
$$T(0, t) = T_0 = \text{constant during expansion} \tag{4}$$

hot end $x = l$:

$$T(l, t) = T_l = \text{constant during compression.}$$

Furthermore, for a steady cyclic regenerator

$$\phi(x, t) = \phi(x, t + \pi) \tag{5}$$

where ϕ stands for any dependent variable and π denotes the time span of a full cycle. It is obvious from equation (2) that the system requires two additional boundary conditions in T_m. They can be easily estimated, however. Also, it will be shown later how they can be conveniently eliminated without any significant loss of accuracy.

The following dimensionless variables will now be introduced:

$$\tau = \frac{T}{T_0}, \quad \tau_m = \frac{T_m}{T_0}, \quad m = \frac{\dot m}{\dot m_0} = \frac{\dot m}{(1/\pi)\oint |\dot m_0| \, dt},$$

$$P = \frac{p}{2(p_H - p_L)}$$

and

$$z = \frac{2(p_H - p_L)V_r x}{\dot m_0 \pi R T_0 l}, \quad \theta = \frac{t}{\pi}. \tag{6}$$

For an ideal gas equations (1) to (3) reduce then to [6]:

Gas energy:

$$\frac{1}{\gamma}\frac{dP}{d\theta} + \frac{\partial}{\partial z}(m\tau) + H(z, \theta)(\tau - \tau_m) = 0 \tag{7}$$

Matrix energy:

$$Q_m \frac{\partial \tau_m}{\partial \theta} - \frac{\partial}{\partial z}\left(K_m \frac{\partial \tau_m}{\partial z}\right) - H(z, \theta)(\tau - \tau_m) = 0 \tag{8}$$

Gas continuity:

$$\frac{\partial m}{\partial z} + \frac{1}{\tau}\frac{dP}{\partial \theta} = 0 \tag{9}$$

where

$$Q_m \equiv \frac{\gamma - 1}{\gamma}\cdot\frac{1 - \epsilon}{\epsilon}\frac{p_m c_m T_0}{2(p_H - p_L)},$$

$$K_m \equiv \frac{\gamma - 1}{\gamma}\cdot\frac{1 - \epsilon}{\epsilon}\frac{\pi k_m T_0}{2(p_H - p_L)\, l^2} z_l^2 \tag{10}$$

$$H \equiv \frac{\gamma - 1}{\gamma}\cdot\frac{\alpha}{\epsilon}\cdot\frac{\pi T_0 K}{2(p_H - p_L)}\cdot\frac{h_t}{K} \equiv C_1\frac{h_t}{K}.$$

In equation (9) the assumption has been invoked that

$$\frac{\partial}{\partial\theta}\left(\frac{P}{\tau}\right) = \frac{1}{\tau}\frac{dP}{d\theta} - \frac{P}{\tau^2}\frac{\partial\tau}{\partial\theta} \cong \frac{1}{\bar{\tau}}\frac{dP}{d\theta} \tag{11}$$

where τ is the gas temperature integrated over a full cycle. A simple order-of-magnitude test shows that this simplification is justified because

$$\frac{1}{\tau}\frac{dP}{d\theta}\Big/\frac{P}{\tau^2}\frac{\partial\tau}{\partial\theta} \cong \frac{\Delta P}{P}\cdot\frac{\tau}{\Delta\tau} \cong \frac{\tau}{\Delta\tau}$$

and for a highly effective regenerator the temperature swing of the gas in time is very small compared to the absolute temperature. However, during a short interval right after flow reversal, $dP/d\theta \cong 0$ and $\partial\tau/\partial\theta \neq 0$, so that equation (11) becomes invalid at this instant. This is the reason for the singularity that appears in the solutions for the gas temperature τ at times of flow reversal (cf. equations (25) and (40)).

The heat transfer coefficient is usually correlated by [1]

$$St\, Pr^{\frac{2}{3}} = a\, Re^n$$

or

$$h_t = K\,|m|^{n_1}\tau^{n_2}$$

where (12)

$$K = a\frac{k_0}{D_h}(\overline{Re_0})^{n_1} Pr^{\frac{1}{3}}, \quad n_1 = 1 + n.$$

The subscript "0" refers to conditions at the cold end, while the term τ^{n_2} accounts for the temperature dependence of the heat transfer coefficient. The values for the constants a and n depend on the matrix and can be found in the literature. For example, Rea and Smith [5] listed $a = 0{\cdot}71$ and $n = -0{\cdot}41$ for a bed of spheres. It is obvious that equation (12) is an approximation since the exact relationship for the time dependence of h_t for a fast cycling flow is not known and cannot correlate the heat transfer coefficient accurately for all situations (e.g. at flow reversal $\dot{m} = 0$ but $h_t > 0$). The correlation for h_t can therefore be adjusted slightly into a more convenient form without any loss of generality. Let

$$h_t = Km^{-n_1-1}\tau^{-n_2}|m|, \tag{13}$$

so that

$$H(z,\theta) = C_1\bar{m}^{n_1-1}\bar{\tau}^{n_2}|m| = \bar{H}(z)\frac{|m|}{\bar{m}} \tag{14}$$

where the average values $\bar{m}$ and $\bar{\tau}$ are defined by

$$\bar{m} = \oint|m|\,d\theta \quad\text{and}\quad \bar{\tau} = \oint\tau\,d\theta. \tag{15}$$

Equations (7)–(9) with $H(z, \theta)$ defined by equation (14) form the system of equations that will be investigated in the following. The pertinent boundary conditions are:

cold end $z = 0$:

$$\begin{aligned} &m(0,\theta) = m_0(\theta) = \dot{m}_0(\theta)/\bar{\dot{m}}_0 \\ &\tau(0,\theta) = 1, \quad \delta \leqslant \theta \leqslant 1 \end{aligned} \tag{16}$$

hot end $z = z_l$:

$$\tau(z_l,\theta) = \tau_l, \quad 0 \leqslant \theta \leqslant \delta$$

where δ is the fraction of π during which the gas is compressed (gas flows from hot to cold end) and $(1 - \delta)$ is the expansion interval (gas flows from cold to hot end). Here the further assumption has been made that flow reversal occurs everywhere in the regenerator at the same instant. This follows readily from equations (7) and (9) in conjunction with the definition for the heat transfer coefficient $H(z, \theta)$ and the assumption for negligible pressure drop.

It is customary to characterize the regenerator performance by its ineffectiveness defined as

$$1 - \eta = \frac{\text{total losses over one cycle}}{\text{ideal heat exchange}}.$$

Losses are due to conduction and the fact that the gas temperature oscillates around the matrix temperature. Thus

$$1 - \eta = \frac{|\oint_{\delta\pi}\dot{m}h\,dt| + (1-\epsilon)A\oint_{\pi} k_m(\partial T_m/\partial x)_l\,dt}{\int_0 |\dot{m}_l|h_l\,dt - \int_{\delta\pi}|\dot{m}_0|h_0\,dt - [\epsilon A_0\int\rho u\,dx]_0^{\delta\pi}}. \tag{17}$$

This expression can be simplified to

$$1 - \eta = 2\frac{|\oint m\tau\,d\theta| + \oint K_m(\partial\tau_m/\partial z)\,d\theta}{\bar{m}_l\tau_l - 1 - z_l/\gamma}. \tag{18}$$

3. SOLUTIONS

In the following four different cases are investigated. The first two deal with an infinitely large matrix heat capacity ($Q_m \to \infty$), first with a particularly simple boundary condition for the mass flux at the cold end, m_0, and then with the general case. These results are exact solutions to the slightly simplified equations (7)–(9). The last two cases treat the finite heat capacity problem ($Q_m < \infty$), first without and then with longitudinal matrix conduction. These results are obtained by the method of perturbations, and can be considered exact throughout the regenerator with the exception of the regions near the boundaries. Throughout the analysis it has been assumed that $Q_m = Q_m(\bar{\tau}_m)$ and $K_m = K_m(\bar{\tau}_m)$ only. Furthermore, the ratio of specific heats, γ, was treated as constant, although the theory is easily extended to a variable γ.

In general, the analyses result in a set of three coupled ordinary differential equations which must be solved numerically. It will be seen, however, that a closed-form solution is possible if conduction is negligible and if the heat transfer coefficient is independent of temperature. Both assumptions appear to be reasonable for many practical applications.

Infinite matrix heat capacity

In many regenerator applications, a gas with a relatively small heat capacity flows through a matrix of very large capacity, resulting in a large value for the parameter Q_m in equation (8). In this case the matrix absorbs the energy transferred to it by the gas easily without any significant temperature change. In the ideal case of an infinitely large heat capacity ($Q_m \to \infty$) the matrix temperature becomes independent of time so that $\tau_m = \tau_m(z)$ only, and equation (8) must be replaced by its cyclic integral

$$\oint H(\tau - \tau_m)\,\mathrm{d}\theta + \frac{\mathrm{d}}{\mathrm{d}z}\left(K_m \frac{\mathrm{d}\tau_m}{\mathrm{d}z}\right) = 0. \qquad (19)$$

It is advantageous to replace the variables m and τ by $\bar{m}$, $\bar{\tau}$, and t, where the perturbation term t is defined as

$$\tau(z, \theta) \equiv \bar{\tau}(z) + t(z, \theta). \qquad (20)$$

It follows from the definition of $\bar{\tau}$ that

$$\oint t\,\mathrm{d}\theta \equiv 0. \qquad (21)$$

To relate $\bar{m}$ to m consider continuity equation (9). Integration results in

$$\bar{\tau}\frac{\mathrm{d}\bar{m}}{\mathrm{d}z} = 1 \qquad (22)$$

and

$$m(z, \theta) = \left(m_0 + \frac{\mathrm{d}P}{\mathrm{d}\theta}\right) - \bar{m}\frac{\mathrm{d}P}{\mathrm{d}\theta}. \qquad (23)$$

A particular simple situation arises if the mass flux at the cold end is such that $m_0 = -\mathrm{d}P/\mathrm{d}\theta$. In reality, of course, this will not necessarily be the case. However, if the volume outside the cold end of the regenerator is constant it obviously represents an excellent approximation as is easily seen from

$$\dot{m}_0 = -\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{pV}{RT}\right\}_{CE} \qquad (24)$$

where the subscript CE denotes properties in the volume outside the cold end of the regenerator. Even for the general case, both functions have some common features indicating that they are similar to some degree. From the assumption of negligible pressure drop and equation (9) it follows that flow reversal occurs everywhere at the same instant, i.e. $m_0 = -\mathrm{d}P/\mathrm{d}\theta = 0$ at times of flow reversal ($\theta = 0, \delta, 1$). Between those times both functions grow in absolute value to some maximum and then diminish again. Furthermore, from equation (23), their integrated values (zeroth moments) are identical, i.e.

$$-\int_0^\delta m_0\,\mathrm{d}\theta = \int_0^\delta \frac{\mathrm{d}P}{\mathrm{d}\theta}\mathrm{d}\theta = \int_\delta^1 m_0\,\mathrm{d}\theta = -\int_\delta^1 \frac{\mathrm{d}P}{\mathrm{d}\theta}\mathrm{d}\theta = \tfrac{1}{2}.$$

In the following the solution for this simple case is presented. It will be shown later that this result is easily extended for $m_0 \neq -\mathrm{d}P/\mathrm{d}\theta$.

If the matrix temperature is constant in time, only changes in mass flux and pressure can cause variation in gas temperature. If $\dot{m}$ and p vary at the same rate, the temperature difference between gas and matrix will be constant at any location during compression and expansion intervals, provided the heat transfer coefficient is of the form of equation (13). Then the solution for τ will have the form

$$\begin{aligned} \tau(z, \theta) &= \bar{\tau}(z) + t_1(z) \qquad 0 < \theta < \delta \\ \tau(z, \theta) &= \bar{\tau}(z) - t_2(z) \qquad \delta < \theta < 1. \end{aligned} \qquad (25)$$

It is obvious that a step in gas temperature at times of flow reversal is physically impossible. The reason for the introduction of this singular point is, of course, the assumption made for equation (9). Introducing (25) reduces equations (7) and (19) to

$$\frac{\gamma - 1}{\gamma} + \bar{m}\frac{\mathrm{d}\bar{\tau}}{\mathrm{d}z} - (2\delta - 1)\frac{\mathrm{d}}{\mathrm{d}z}\left(K_m \frac{\mathrm{d}\tau_m}{\mathrm{d}z}\right) = C_1 \bar{m}^{n_1 - 1}\bar{\tau}^{n_2}\left[C_2 - K_m \frac{\mathrm{d}\tau_m}{\mathrm{d}z}\right] \qquad (26)$$

$$(2\delta - 1)\left\{\frac{\gamma - 1}{\gamma} + \bar{m}\frac{\mathrm{d}\bar{\tau}}{\mathrm{d}z}\right\} - 4\delta(1 - \delta)\frac{\mathrm{d}}{\mathrm{d}z}\left(K_m \frac{\mathrm{d}\tau_m}{\mathrm{d}z}\right) = C_1 \bar{m}^{n_1}\bar{\tau}^{n}(\bar{\tau} - \bar{\tau}_m) \qquad (27)$$

$$2\delta t_1 = 2(1 - \delta)t_2 = 4\delta(1 - \delta)\frac{1}{\bar{m}}\left[C_2 - K_m \frac{\mathrm{d}\tau_m}{\mathrm{d}z}\right] \qquad (28)$$

where C_2 is an integration constant to be determined by the boundary conditions. Usually $C_1 \gg 1$ and $|2\delta - 1| \ll 1$, so that the second-order term in equation (26) can be dropped. Furthermore, in a highly efficient regenerator $|\bar{\tau} - \tau_m| \ll 1$ or $\mathrm{d}\tau_m/\mathrm{d}z \cong \mathrm{d}\bar{\tau}/\mathrm{d}z$. Then

$$\frac{\gamma - 1}{\gamma} = \bar{m}\frac{\mathrm{d}\bar{\tau}}{\mathrm{d}z} \cong C_1\bar{m}^{n_1 - 1}\bar{\tau}^{n_2}\left[C_2 - K_m\frac{\mathrm{d}\bar{\tau}}{\mathrm{d}z}\right] \tag{29}$$

which, coupled with equation (22), must in general be solved numerically, subject to the boundary conditions

cold end $z = 0$:

$$\bar{m} = 1 \tag{30}$$

$$\bar{\tau} = 1 + t_2 = 1 + 2\delta\left[C_2 - K_m\frac{\mathrm{d}\bar{\tau}}{\mathrm{d}z}\right] \tag{31}$$

hot end $z = z_l$:

$$\bar{\tau} = \tau_l - t_l = \tau_l - 2(1 - \delta)\left[C_2 - K_m\frac{\mathrm{d}\bar{\tau}}{\mathrm{d}z}\right]. \tag{32}$$

Thus the solution for τ_m has been decoupled from the solution for $\bar{\tau}$ and $\bar{m}$. Once $\bar{\tau}$ and $\bar{m}$ are known, τ_m is readily calculated from equation (25). Again the derivative in τ_m can be replaced by a derivative in $\bar{\tau}$, thus avoiding to estimate boundary conditions for the matrix temperature.

The regenerator ineffectiveness follows from equation (18) to be

$$1 - \eta = \frac{2C_2}{\bar{m}_l\tau_l - 1 - z_l/\gamma}. \tag{33}$$

If $m_0 \neq -\mathrm{d}P/\mathrm{d}\theta$ the solution is similar. A second perturbation term $\vartheta(z, \theta)$ is added to equation (25) so that

$$\begin{aligned} \tau(z,\theta) &\equiv \bar{\tau}'(z) + t_1(z) + \vartheta(z, 0) \qquad 0 < \theta < \delta \\ \tau(z,\theta) &\equiv \bar{\tau}'(z) - t_2(z) + \vartheta(z, \theta) \qquad \delta < \theta < 1 \end{aligned} \tag{34}$$

where $\bar{\tau}(z) \equiv \oint(\tau - \vartheta)\,\mathrm{d}\theta \equiv \bar{\tau} - \bar{\vartheta}$. As $|\bar{\vartheta}| \ll \bar{\tau}$ one can replace $\bar{\tau}$ by $\bar{\tau}'$ in equations (9) and (13). Equations (22), (23) and (26)–(33) then continue to hold (with $\bar{\tau}'$ instead of $\bar{\tau}$), while ϑ turns out to be

$$\vartheta(z,\theta) = -\left(1 + \frac{\bar{m}}{m}\frac{\mathrm{d}P}{\mathrm{d}\theta}\right)\int_z^{z_l}\exp\left\{-\int_z^{\eta} C_1\bar{m}^{n_1 - 1}\bar{\tau}'^{n_2}\mathrm{d}\xi\right\} \times\left[\frac{\gamma - 1}{\gamma} + \frac{t_1}{\bar{\tau}'}\right]\frac{\mathrm{d}\eta}{\bar{m}}, \qquad 0 < \theta < \delta$$

$$\vartheta(z,\theta) = \left(1 + \frac{\bar{m}}{m}\frac{\mathrm{d}P}{\mathrm{d}\theta}\right)\int_0^{z}\exp\left\{-\int_0^{\eta} C_1\bar{m}^{n_1 - 1}\bar{\tau}'^{n_2}\,\mathrm{d}\xi\right\} \times\left[\frac{\gamma - 1}{\gamma} - \frac{t_2}{\bar{\tau}'}\right]\frac{\mathrm{d}\eta}{\bar{m}}, \qquad \delta < \theta < 1. \tag{35}$$

Finite matrix heat capacity

In many applications, the matrix heat capacity is not large enough to render the matrix temperature τ_m a function of position only, i.e. the temperature fluctuation of the matrix is such that it cannot be neglected. It is, therefore, of importance to determine how the results of the preceding section are affected by a finite Q_m. First the somewhat simpler case of negligible matrix heat conduction will be treated. These results will then later be extended to include conduction effects.

For the analysis, the temperatures will be broken up according to equation (20), so that

$$\tau(z, \theta) = \bar{\tau}(z) + t(z, \theta) \tag{20}$$

and

$$\tau_m(z, \theta) = \bar{\tau}_m(z) + t_m(z, \theta). \tag{36}$$

With this definition and $K_m \equiv 0$ equations (7) and (8) can be written as

$$\frac{1}{\gamma}\frac{\mathrm{d}P}{\mathrm{d}\theta} + \frac{\partial}{\partial z}(m\bar{\tau}) + Q_m\frac{\partial t_m}{\partial\theta} = 0 \tag{37}$$

and

$$Q_m\frac{\partial t_m}{\partial \theta} - C_1\bar{m}^{n_1 - 1}\bar{\tau}^{n_2}|m|(\bar{\tau} - \bar{\tau}_m + t - t_m) = 0, \tag{38}$$

where the term $\partial(mt)/\partial z$ in equation (37) has been dropped as it is small compared to $\partial(m\bar{\tau})/\partial z$. However, neglecting this term reduces the order of t. Therefore, while equations (37) and (38) describe the regenerator performance accurately throughout the interior of the regenerator, this might not be true very close to the boundaries, as the fluctuation boundary condition for t cannot be satisfied.

The solutions to this system of equations are:

$$m(z, \theta) = \frac{\mathrm{d}q}{\mathrm{d}\theta} - \bar{m}\frac{\mathrm{d}p}{\mathrm{d}\theta} \tag{23}$$

$$t_m(z,\theta) = \left\{\frac{\gamma - 1}{\gamma} + \bar{m}\frac{\mathrm{d}\bar{\tau}}{\mathrm{d}z}\right\}\frac{P - \bar{P}}{Q_m} - \frac{\mathrm{d}\bar{\tau}}{\mathrm{d}z}\frac{q - \bar{q}}{Q_m} \tag{39}$$

$$t(z,\theta) = \left\{\frac{\gamma - 1}{\gamma} + \bar{m}\frac{\mathrm{d}\bar{\tau}}{\mathrm{d}z}\right\}\left\{\frac{P - \bar{P}}{Q_m} + \frac{1 - 2\delta \pm 1}{c_2\bar{m}^{n_1}\bar{\tau}^{n_2}}\right\} - \frac{\mathrm{d}\bar{\tau}}{\mathrm{d}z}\frac{q - \bar{q}}{Q_m} + \frac{\gamma - 1}{\gamma}\left\{\frac{1}{|m|}\frac{\mathrm{d}q}{\mathrm{d}\theta} - \left(\overline{\frac{1}{|m|}\frac{\mathrm{d}q}{\mathrm{d}\theta}}\right)\right\} \tag{40}$$

where the abbreviation

$$q(\theta) = \int_0^\theta \left(m_0 + \frac{dP}{d\theta}\right) d\theta$$

has been used. The upper sign in equation (40) is valid during compression ($0 < \theta < \delta$), the lower sign during expansion ($\delta < \theta < 1$). The averaged properties must be found from the ordinary differential equations

$$\bar{\tau}\frac{d\bar{m}}{dz} = 1 \tag{22}$$

$$\frac{\gamma - 1}{\gamma} + \bar{m}\frac{d\bar{\tau}}{dz} = c_1 \bar{m}^{n_1 - 1}\bar{\tau}^{n_2} \times \left\{C_2 + \frac{\gamma - 1}{\gamma}\frac{\overline{(m_0 P)}}{Q_m}\right\} \tag{41}$$

$$(2\delta - 1)\left\{\frac{\gamma - 1}{\gamma} + \bar{m}\frac{d\bar{\tau}}{dz}\right\} = C_1 \bar{m}^{n_1}\bar{\tau}^{n_2}(\bar{\tau} - \bar{\tau}_m) - \frac{\gamma - 1}{\gamma}\overline{\left(\frac{1}{|m|}\frac{dq}{d\theta}\right)} \tag{42}$$

subject to the boundary conditions

cold end $z = 0$:

$$\bar{m} = 1 \tag{43}$$

$$\bar{\tau} = 1 - \frac{1}{1 - \delta}\int_\delta^1 t\, d\theta \cong 1 + 2\delta\left[C_2 + \frac{\gamma - 1}{\gamma}\frac{\overline{(m_0 P)}}{Q_m}\right] \tag{44}$$

hot end $z = z_l$:

$$\bar{\tau} = \tau_l - \frac{1}{\delta}\int_0^\delta t\, d\theta \cong \tau_l - 2(1 - \delta) \times \left[C_2 + \frac{\gamma - 1}{\gamma}\frac{\overline{(m_0 P)}}{Q_m}\right]. \tag{45}$$

For the evaluation of the regenerator effectiveness equation (33) remains valid.

To include the effects of matrix conduction, the approach is similar to the one described earlier. Equations (20) and (36) are redefined as

$$\tau(z, \theta) = \bar{\tau}(z) + t(z, \theta) + \vartheta(z, \theta) \tag{46}$$

$$\tau_m(z, \theta) = \bar{\tau}_m(z) + t_m(z, \theta) + \vartheta_m(z, \theta). \tag{47}$$

Here t and t_m have been evaluated above and describe the major perturbations of τ and τ_m caused by mass flux and pressure variations, and the ϑ and ϑ_m are minor perturbations due to matrix conduction effects.

Thus from comparing equation (7) with (37) and (8) with (38), respectively:

$$\frac{\partial}{\partial z}(mt) + Q_m\frac{\partial \vartheta_m}{\partial \theta} - \frac{d}{dz}\left(K_m\frac{d\bar{\tau}_m}{dz}\right) = 0 \tag{48}$$

and

$$Q_m\frac{\partial \vartheta_m}{\partial \theta} - C_1 \bar{m}^{n_1 - 1}\bar{\tau}^{n_2}|m|(\vartheta - \vartheta_m) - \frac{d}{dz}\left(K_m\frac{d\bar{\tau}_m}{dz}\right) = 0 \tag{49}$$

where the terms $\partial(m\vartheta)/\partial z$ and $\partial(t_m + \vartheta_m)/\partial z$ have been neglected, as they are anticipated to be small compared to $\partial/\partial z(mt)$ and $d\bar{\tau}_m/dz$.

The results are

$$\vartheta_m(z, \theta) = \frac{1}{Q_m}\left\{\frac{d}{dz}\overline{(mt)} (\theta - \tfrac{1}{2}) - \frac{\partial}{\partial z}\left(\int_0^\theta mt\, d\theta - \oint \int_0^\theta mt\, d\theta\, d\theta\right)\right\} \tag{50}$$

$$\vartheta(z, \theta) = \vartheta_m(z, \theta) - \frac{1}{C_1 \bar{m}^{n_1}\bar{\tau}^{n_2}}\frac{\bar{m}}{|m|}\frac{\partial}{\partial z}(mt), \tag{51}$$

while equations (41) and (42) must be replaced by

$$\frac{\gamma - 1}{\gamma} + \bar{m}\frac{d\bar{\tau}}{dz} = C_1 \bar{m}^{n_1 - 1}\bar{\tau}^{n_2} \times \left\{C_2 - K_m\frac{d\bar{\tau}_m}{dz} + \frac{\gamma - 1}{\gamma}\frac{\overline{(m_0 P)}}{Q_m}\right\} \tag{52}$$

and

$$(2\delta - 1)\left\{\frac{\gamma - 1}{\gamma} + \bar{m}\frac{d\bar{\tau}}{dz}\right\} = C_1 \bar{m}^{n_1}\bar{\tau}^{n_2}(\bar{\tau} - \bar{\tau}_m) - \frac{\gamma - 1}{\gamma}\overline{\left(\frac{1}{|m|}\frac{dq}{d\theta}\right)} + \overline{\left(\frac{\bar{m}}{|m|}\frac{\partial}{\partial z}(mt)\right)} \tag{53}$$

subject to the boundary conditions

cold end $z = 0$:

$$\bar{m} = 1 \tag{54}$$

$$\bar{\tau} \cong 1 - \frac{1}{1 - \delta}\int_\delta^1 (t + \vartheta)\, d\theta \cong 1 + 2\delta\left[C_2 - K_m\frac{d\bar{\tau}_m}{dz} + \frac{\gamma - 1}{\gamma}\frac{\overline{(m_0 P)}}{Q_m}\right] \tag{55}$$

hot end $z = z_l$:

$$\bar{\tau} \cong \tau_l - \frac{1}{\delta}\int_0^{\delta}(t + \vartheta)\,\mathrm{d}\theta \cong 1 - 2(1 - \delta)$$

$$\times \left[C_2 - K_m \frac{\mathrm{d}\bar{\tau}_m}{\mathrm{d}z} + \frac{\gamma - 1}{\gamma}\frac{\overline{(m_0 P)}}{Q_m}\right]. \tag{56}$$

Again $\mathrm{d}\bar{\tau}_m/\mathrm{d}z$ can be approximated by $\mathrm{d}\bar{\tau}/\mathrm{d}z$ in equation (52) in order to decouple equation (53) from equations (22) and (52). The average temperature of the matrix, $\bar{\tau}_m$, can then be calculated from equation (53), although the calculation of the last two terms in that equation will be rather tedious. However, a simple order-of-magnitude test reveals that

$$\left(\frac{1}{|m|}\frac{\mathrm{d}q}{\mathrm{d}\theta}\right) \ll 1.$$

Furthermore, if $\mathrm{d}q/\mathrm{d}\theta \equiv 0$, the last term reduces to

$$\left(\overline{\frac{m}{|m|}\frac{\partial}{\partial z}(mt)}\right) = -4\delta(1 - \delta)\frac{\mathrm{d}}{\mathrm{d}z}\left(K_m \frac{\mathrm{d}\bar{\tau}_m}{\mathrm{d}z}\right) \tag{57}$$

if the pressure is such that the average pressures for both half-cycles are equal. Using this, equation (53) reduces to equation (27).

Again, the expression for the regenerator ineffectiveness does not change, so that equation (33) remains valid also for this case.

Explicit analytical solution for negligible conduction

It has been shown how the partial differential equations governing the performance of thermal regenerators can be reduced to ordinary differential equations. In general, these equations are still so complex that they can only be solved numerically. There exists, however, an important special case when an explicit analytical solution is possible. This is the case if longitudinal matrix conduction, as well as the temperature dependence of the heat transfer coefficient, is negligible, i.e. if $K_m \equiv n_2 \equiv 0$, which is approximately true for moderate temperatures.

Consider equations (26) or (41). If $K_m \cong 0$ and $\bar{\tau}^{n_2} \cong 1$ they can be rewritten as

$$\frac{\gamma - 1}{\gamma} + \bar{m}\frac{\mathrm{d}\bar{\tau}}{\mathrm{d}z} = C\bar{m}^n \tag{58}$$

where

$$n = n_1' - 1 \quad \text{and} \quad C = C_1 C_2$$

or

$$C_1\left\{C_2 + \frac{\gamma - 1}{\gamma}\frac{\overline{(m_0 P)}}{Q_m}\right\},$$

assuming that under these conditions also the matrix heat capacity can be treated as constant. This equation must be solved in combination with

$$\bar{\tau}\frac{\mathrm{d}\bar{m}}{\mathrm{d}z} = 1 \tag{22}$$

subject to the boundary conditions

cold end $z = 0$:

$$\bar{m} = 1 \qquad \bar{\tau} = 1 \tag{59}$$

hot end $z = z_l$:

$$\bar{\tau} = \tau_l,$$

which have been simplified by dropping the small terms for the sake of clarity.

The axial distance is easily eliminated from equations (22) and (58) to yield

$$\bar{\tau} = \bar{m}^{-(\gamma-1)/\gamma}\exp\left\{\frac{C}{n}(1 - \bar{m}^n)\right\} \qquad n \neq 0 \tag{60}$$

$$\bar{\tau} = \bar{m}^{C-(\gamma-1)/\gamma} \qquad n = 0. \tag{61}$$

With this expression equation (22) can be solved for the average mass flux $\bar{m}$ and gas temperature $\bar{\tau}$ as functions of longitudinal distance z to give

$$\bar{m} = \left\{\frac{1}{\beta}M_\nu^{-1}[M_\nu(\beta) - n\beta^{1-\nu}\,\mathrm{e}^{-\beta}z]\right\}^{1/n} \qquad n \neq 0 \tag{62}$$

$$\bar{\tau} = \left\{\frac{1}{\beta}M_\nu^{-1}[M_\nu(\beta) - n\beta^{1-\nu}\,\mathrm{e}^{-\beta}z]\right\}^{-(\gamma-1)/n\gamma}$$

$$\times \exp\{\beta - M_\nu^{-1}[M_\gamma(\beta) - n\beta^{1-\nu}\,\mathrm{e}^{-\beta}z]\} \qquad n \neq 0 \tag{63}$$

and

$$\bar{m} = \left\{1 + \left[C + 1 - \frac{\gamma - 1}{\gamma}\right]z\right\}^{\frac{1}{C+1-(\gamma-1)/\gamma}} \qquad n = 0 \tag{64}$$

$$\bar{\tau} = \left\{1 + \left[C + 1 - \frac{\gamma - 1}{\gamma}\right]z\right\}^{\frac{C-(\gamma-1)/\gamma}{C+1-(\gamma-1)/\gamma}} \qquad n = 0. \tag{65}$$

Here the abbreviations β and ν have been introduced as

$$\beta = -C/n \quad \text{and} \quad \nu = \frac{1}{n}\cdot\frac{\gamma - 1}{\gamma} + \frac{n - 1}{n} \tag{66}$$

while the incomplete gamma function $M_\nu(\beta)$ has been defined to be

$$M_\nu(\beta) \equiv \int_\beta^\infty \eta^{-\nu} e^{-\eta} d\eta. \quad (67)$$

In equations (62)–(65) the first two boundary conditions have been employed. The last one is needed to evaluate the constant β (or C, C_2), which must be done by trial and error. While some tabulations of the incomplete gamma functions can be found in the literature [7], an extensive listing for values of ν relevant for regenerator applications has also been given [8].

Comparison with exact solution

A computer program for the exact numerical solution of the general equations (1)–(3) without the simplifications (11) and (13) has been developed to test the theories presented in the previous sections. In particular, it will be seen what influence the omission of the term $[(P/\tau^2)(\partial\tau/\partial\theta)]$ in equation (9) exerts on the solution, and whether, for finite Q_m, dropping $\partial(mt)/\partial z$ in equation (37) really only affects the regions close to the boundaries.

Due to the extremely unfavorable boundary conditions, only few approaches to a numerical solution are conceivable. The one employed here approximates time derivatives by backward differences, while directional derivatives are written as forward differences during compression ($0 < \theta \leq \delta$) and backward differences during expansion ($\delta < \theta \leq 1$). An initial distribution of temperatures and mass flow is guessed for a certain time (for example, by the methods described earlier in this paper) and new values are calculated time step after time step. Physically this method calculates the cool-down of the generator from the initial (or approximate) temperature distribution until the quasi-steady state is reached. Obviously, a accurate initial distribution is crucial. Usually, the cool-down time for a compact refrigerator operating according to the Stirling or Gifford–McMahon cycles is about one to two hours. Thus, if an accurate initial profile is employed, cool-down time can be expected to be of the order of a few minutes. Therefore, a few hundred iterations will usually be necessary, and somewhere between 1 and 5 min of computer time will be needed, even with a high-speed computer like the CDC 6400, which was used here at Berekeley.

To test the previously derived theories this computer program was employed for the special case of

$$\delta = \tfrac{1}{2}, \quad K_m = 0 \quad \text{and} \quad m_0 = -\frac{dP}{d\theta}$$

with

$$C_1 = Q_m = 100, \quad z_l = 1 \quad \text{and} \quad \tau_l = 5.$$

Two different pressure histories were investigated:

(i) linear ("saw-tooth"):

$$P = P_L + \theta, \quad 0 \leq \theta \leq \tfrac{1}{2}$$

$$P = P_L + (1 - \theta), \quad \tfrac{1}{2} \leq \theta \leq 1$$

(ii) sinusoidal:

$$P = \bar{P} + \tfrac{1}{4}\cos 2\pi\theta, \quad 0 \leq \theta \leq 1.$$

If the average values $\bar{m}$ and $\bar{\tau} = \bar{\tau}_m$ are evaluated from equations (22) and (41), it was found that they practically coincide with the ones found by the numerical solution (deviation $<\frac{1}{2}$ per cent). Figures 1–3 demonstrate how equations (39) and (40) compare to the exact numerical solution for case (i), Figs. 4–6 for case (ii).

First consider Fig. 1, which depicts the cold end ($z = 0$) for case (i). As expected, agreement is not too good during expansion as the boundary condition for t had been neglected in the analysis. However, agreement for the compression half-cycle is excellent. This figure also demonstrates how the singularity in equation (40) which was introduced by neglecting $[(P/\tau^2)(\partial\tau/\partial\theta)]$, is overcome by the numerical solution. To show how minute the influence of the neglected boundary condition is, the temperature histories have been plotted for a point inside the matrix—but still close to the boundary—at $x/l = 0{\cdot}1$. Figure 2 shows the excellent agreement between theory and exact solution, the only deviation being at times close to flow reversal. The same is true for all other locations throughout the regenerator. Figure 3 shows the time history of the mass flow at the location $x/l = 0{\cdot}1$. Obviously, what has been said about the temperatures is also true for the mass flow.

Now consider Fig. 4, depicting time histories at the cold end for case (ii). While agreement for the matrix temperature is fairly good, this is not true for the gas temperature (except for the fact that values integrated over half-cycles more or less coincide, ensuring agreement for the solution of average properties). The reason for this deviation is that different definitions for the heat transfer coefficient have been used. To prove this, introduce the original definition for h_t in equation (38). Then

$$Q_m \frac{\partial t_m}{\partial \theta} - C_1 \bar{m}^{n_1} \bar{\tau}^{n_2} \left|\frac{dP}{d\theta}\right|^{n_1} (t - t_m) = 0 \quad (68)$$

where $\bar{\tau} = \bar{\tau}_m$ and $m_0 = -dP/d\theta$ have been used. Thus

$$t(z, \theta) = t_m(z, \theta) + \frac{Q_m}{C_1 \bar{m}^{n_1} \bar{\tau}^{n_2}} \left|\frac{dP}{d\theta}\right|^{-n_1} \frac{\partial t_m}{\partial \theta} \quad (69)$$

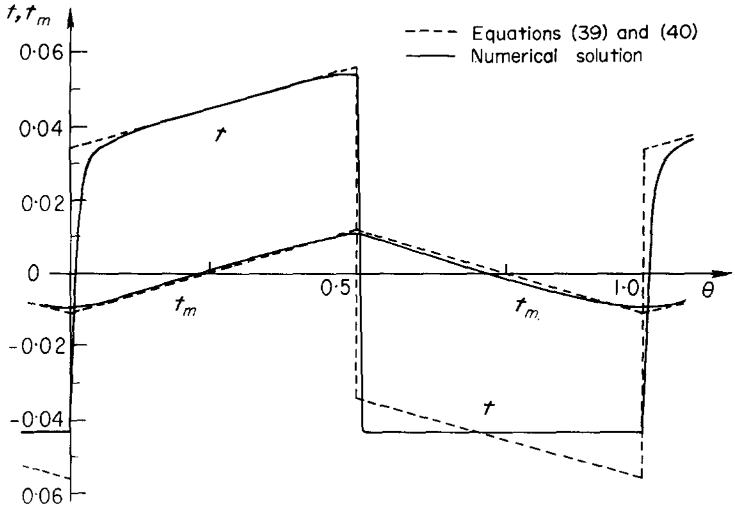

FIG. 1. Comparison of theoretical and exact temperature fluctuations at $x/l = 0$ (linear pressure variation).

or, using equations (39) and (41):

$$t(z, \theta) = t_m(z, \theta) + \frac{C_2}{\bar{m}} \left|\frac{dP}{d\theta}\right|^{-n_1} \frac{dP}{d\theta}. \tag{70}$$

If the pressure is linear in time $|dP/d\theta| \equiv 1$ and equation (40) follows. However, if $dP/d\theta \neq 1$ the effect on the gas temperature can be considerable, as seen from Fig. 4 and 5 where plots of equation (70) have been added for sinusoidal pressure variation. Their agreement with the exact solution is again excellent. Figure 6, finally, shows how little the mass flow has been affected by the simplifications also for this case.

4. CONCLUDING REMARKS

The analyses presented above demonstrate how the equations governing the performance of cyclic regenerators can be solved conveniently and accurately. They also make it possible to draw a number of valuable conclusions on the influence of physical parameters, even without going through a detailed calculation.

Consider first the relative duration of compression and expansion. The symbol δ had been defined as the fraction of a full cycle during which compression takes place. It is immediately obvious from equations (29) and (52) that the exact value for δ has only a minute

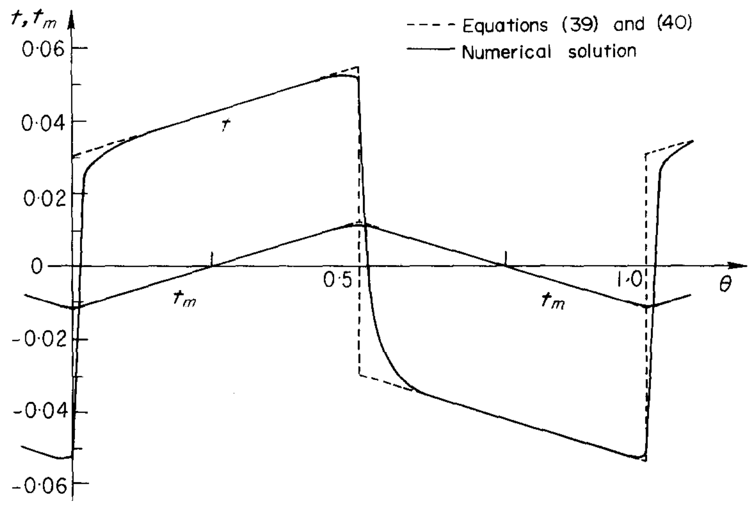

FIG. 2. Comparison of theoretical and exact temperature fluctuations at $x/l = 0·1$ (linear pressure variation).

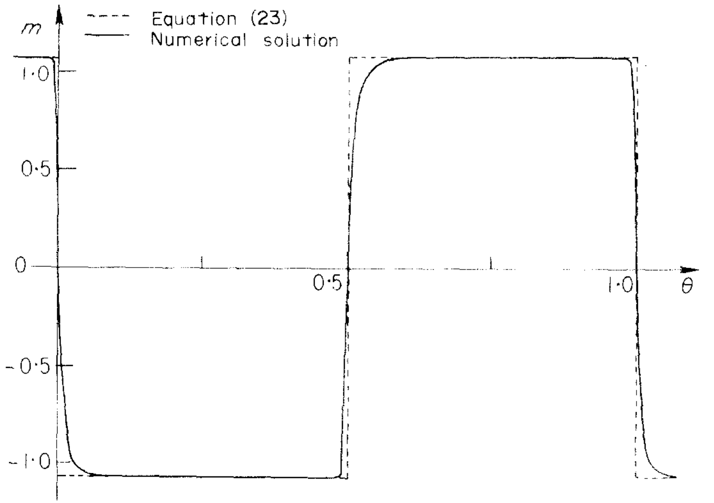

FIG. 3. Comparison of theoretical and exact mass flux distributions at $x/l = 0{\cdot}1$ (linear pressure variation).

influence on the mean gas temperature and mean mass flux through their boundary conditions. Furthermore, the constant C_2 is evaluated from these equations. From this it follows that the relative duration of compression and expansion has also only insignificant influence on the regenerator effectiveness. There is, however, some effect on the average matrix temperature (of the order of $(2\delta - 1)\,t$) as seen from equations (27) and (53) and on the perturbations, t and t_m. However, it is imperative to keep in mind equation (11), i.e. that the duration of flow reversal is small compared to the duration of a half-cycle. The above conclusions do not hold for the cases $\delta \simeq 0$ and $\delta \simeq 1$, which rarely occur in practical situations.

The distribution of the mass flux at the cold end appears to be equally unimportant. In most practical situations m_0 will be so that $|dq/d\theta| \ll |dP/d\theta|$ even if $m_0 \neq -dP/d\theta$. If m_0 is such that $dq/d\theta$ becomes substantial there will be some influence on the regenerator performance, however small. The average gas temperature and mass flow will be affected by (m_0P) only if Q_m varies rapidly with temperatures. Even then the influence will be negligible as $|m_0P| \ll 1((m_0P) = 0$ if $m_0 = -dP/d\theta)$. It is difficult to estimate the impact on the average matrix temperature. In general, however, also $\bar{t}_m$ will be little

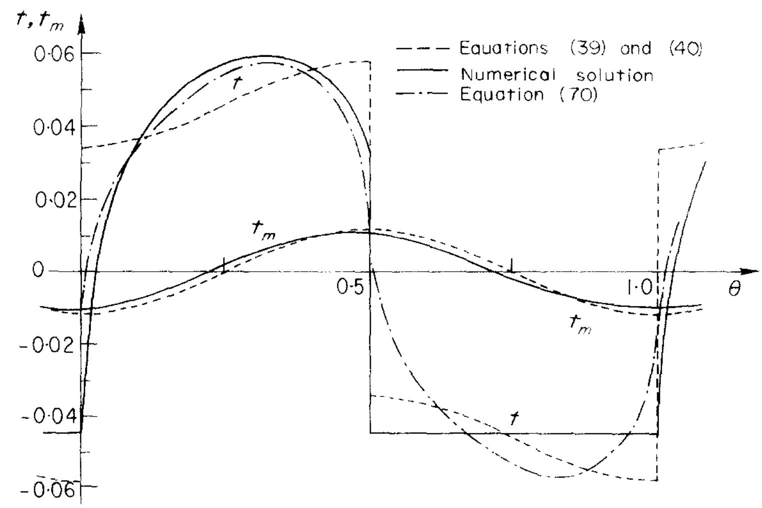

FIG. 4. Comparison of theoretical and exact temperature fluctuations at $x/l = 0$ (sinusoidal pressure variation).

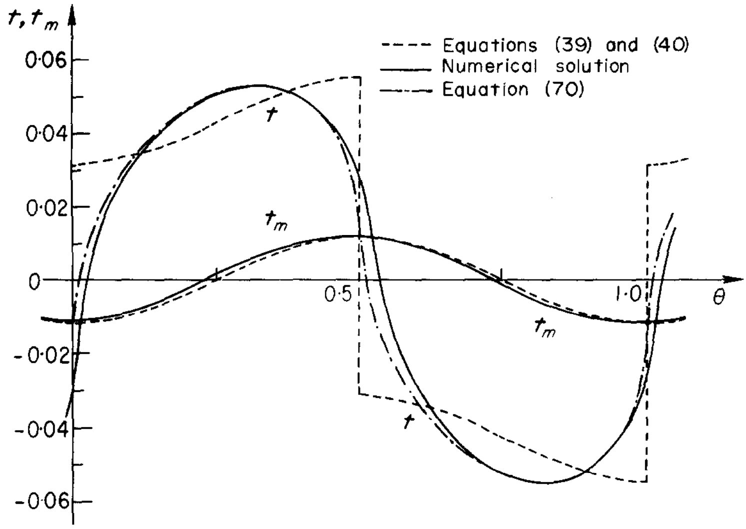

FIG. 5. Comparison of theoretical and exact temperature fluctuations at $x/l = 0{\cdot}1$ (sinusoidal pressure variation).

changed as

$$\left|\frac{1}{|m|}\frac{\mathrm{d}q}{\mathrm{d}\theta}\right| \ll 1.$$

The most surprising finding in the present analysis is that the relative magnitude of the matrix heat capacity turns out to be irrelevant. If $m_0 \equiv -\mathrm{d}P/\mathrm{d}\theta$, the value for Q_m displays no influence whatsoever on average properties and effectiveness. If $m_0 \neq -\mathrm{d}P/\mathrm{d}\theta$ and $Q_m < \infty = \text{const.}$, the average properties are again uneffected while the effectiveness can be calculated from

$$\frac{(1-\eta)_{Q_m}}{(1-\eta)_\infty} = 1 - \frac{\gamma - 1}{\gamma}\cdot\frac{(\overline{m_0 P})}{Q_m \cdot C_2(Q_m \to \infty)}. \tag{71}$$

It appears therefore that a positive value of $(\overline{m_0 P})$ can enhance the regenerator effectiveness. This effect will be small, however, as $|\overline{m_0 P}| \ll 1$. On the other hand, there will be, of course, a substantial influence on the matrix temperature perturbation t_m, and hence on t, as its amplitude is inversely proportional to the heat capacity parameter Q_m.

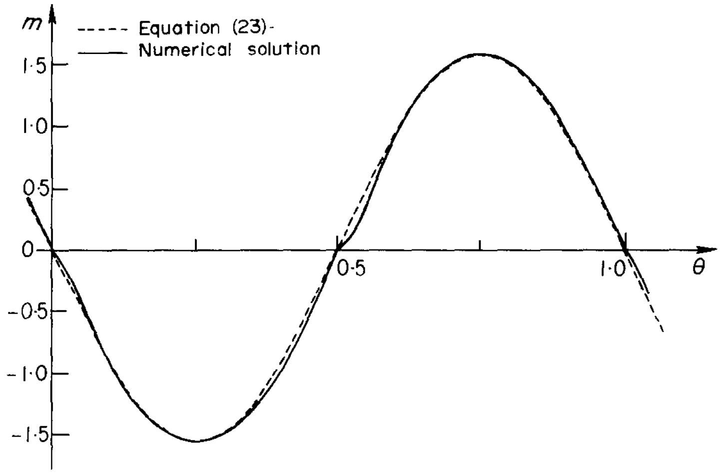

FIG. 6. Comparison of theoretical and exact mass flux distributions at $x/l = 0{\cdot}1$ (sinusoidal pressure variation).

It should be emphasized that the above conclusions for the finite heat capacity case apply only to the "rapidly cycling" regenerator, i.e. when

$$|\partial(mt)/\partial z|/|\partial(m\bar{\tau})/\partial z| \ll 1$$

or equivalently $|t|/\bar{\tau} \ll 1$. From equation (40), it can be estimated that

$$|t|/\bar{\tau} = 0\left\{\left(\frac{\sigma-1}{\sigma}+\bar{m}\frac{\mathrm{d}\bar{\tau}}{\mathrm{d}z}\right)\left(\frac{1}{4Q_m}+\frac{1}{C_1}\right)\frac{1}{\bar{\tau}}\right\}$$

$$\leqslant 2\left(\frac{\tau_l-1}{z_l}\right)\left(\frac{1}{4Q_m}+\frac{1}{C_1}\right).$$

Let $|t/\bar{\tau} < 0{\cdot}1$, it follows

$$\frac{1}{Q_m} \leqslant \frac{z_l}{5(\tau_l - 1)} - \frac{4}{C_1}. \qquad (72)$$

This condition is satisfied in many actual operating situations. For instance, Figs. 1–6 ($C_1 = Q_m = 100$, $z_l = 1$, $\tau_l = 5$) demonstrate that (72) is indeed very conservative. For extreme cases with the combination of sufficiently large blow time, low temperature and small heat capacity, the present analysis and conclusions do not hold.

Acknowledgements—The study was conducted under the support of the Lockheed Missiles and Space Company, and many helpful discussions with Dr. A. P. Glassford of Lockheed are particularly appreciated.

REFERENCES

1. W. M. Kays and A. L. London, *Compact Heat Exchangers*. McGraw-Hill, New York (1964).
2. B. H. Schultz, Regenerators with longitudinal heat conduction, General discussion on heat transfer, Inst. of Mech. Engrs and ASME (1951).
3. G. D. Bahnke and C. P. Howard, The effect of longitudinal heat conduction on periodic-flow heat exchanger performance, *J. Engng Ind.* **86A**, 105–120 (1964).
4. P. A. Rios and J. L. Smith, Jr., The effect of variable specific heat of the matrix on the performance of thermal regenerators, *Adv. Cryogen. Engng* **13**, 566–573 (1967).
5. S. N. Rea and J. L. Smith, Jr., The influence of pressure cycling on thermal regenerators, *J. Engng Ind.* **89B**, 563–569 (1967).
6. M. F. Modest and C. L. Tien, Analysis of real-gas and matrix-conduction effects in cyclic cryogenic regenerators, *J. Heat Transfer* **95C**, 199–205 (1973).
7. W. Abramowitz and I. A. Stegun (Eds.), *Handbook of Mathematical Functions*, National Bureau of Standards, Applied Mathematics Series 55 (1964).
8. M. F. Modest, Theoretical analysis of the performance of cryogenic thermal regenerators subjected to rapidly cycling flows, Ph.D. Dissertation, Department of Mechanical Engineering, University of California, Berkeley, California (1972).

ETUDE THERMIQUE DE REGENERATEURS CRYOGENIQUES CYCLIQUES

Resume—On établit le système d'équations aux dérivées partielles associées à leurs conditions aux limites pour préciser les performances des régénérateurs soumis à des écoulements cycliques en tenant compte d'effets généralement négligés comme les changements d'énergie interne du fluide dus aux pressions cycliques et à la conduction longitudinale. On obtient des solutions exactes dans le cas d'une capacité thermique de matrice infiniment grande. Dans le cas de capacité thermique de matrice finie, la méthode des perturbations est utilisée et la solution peut être considérée comme exacte dans le régénérateur sauf dans les régions proche des limites. Les résultats sont contenus en général dans un système de trois équations différentielles couplées qui peut être résolu numériquement. On présente néanmoins une solution analytique pour le cas important d'une conduction thermique de matrice négligeable.

THERMISCHE ANALYSE VON ZYKLISCH KÄLTEERZEUGENDEN REGENERATOREN

Zusammenfassung—Um das Verhalten von Regeneratoren, die periodisch wechselnden Strömungen ausgesetzt sind, zu beschreiben, wurde ein System von partiellen Differentialgleichungen mit ihren Randbedingungen aufgestellt, die sonst allgemein vernachlässigte Effekte, wie z.B. den inneren Energieaustausch des Fluids infolge periodisch wechselnder Drucke und longitudinale Matrix-Leitung berücksichtigen.

Für den Fall einer unendlich grossen Matrix-Wärmekapazität wurde eine exakte Lösung gefunden. Für den Fall einer endlichen Matrix-Wärmekapazität führt die Störungsmethode zu einer Lösung, die exakt für einen Regenerator ausschliesslich der Randzonen gilt. Die Ergebnisse sind in einem Satz von drei gekoppelten einfachen Differentialgleichungen enthalten, die numerisch gelöst werden müssen. Für den wichtigen Fall der vernachlässigbaren Matrix-Wärmeleitung ist die Lösung in geschlossener Form dargestellt.

ТЕРМИЧЕСКИЙ АНАЛИЗ ЦИКЛИЧЕСКИХ КРИОГЕННЫХ РЕГЕНЕРАТОРОВ

Аннотация—Получена система дифференциальных уравнений в частных производных с граничными условиями для описания характеристик циклических регенераторов. В ней учитываются такие эффекты, которыми обычно пренебрегают, а именно : изменение внутренней энергии жидкости под влиянием периодических изменений давления и продольная теплопроводность матрицы. Точные решения получены для случая бесконечно большой теплоемкости матрицы. Для случая конечной теплоемкости матрицы используется метод возмущений. Полученные решения можно считать точными для всего регенератора за исключением граничных областей. В общем, результаты объединяются в систему из трех связанных обыкновенных дифференциальных уравнений, решаемую численно. Однако, для важного случая пренебрежимо малой теплопроводности дается аналитическое решение.